Fourier form of the dressing method: simple example of integrable (2+1)-dimensional integral-differential equation

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Abstract

We represent the Fourier form of the dressing method, which is effective for construction of multidimensional integral-differential equations together with their solutions. Example of integrable (but non-physical) expansion of Intermediate Long Wave equation in (2+1)-dimensions is considered.

1   Introduction

Dressing method represents a productive tool for solving large family of nonlinear equations of Mathematical Physics. Many equations from this family have well known applications in different branches of mathematical Physics: hydrodynamics, plasma physics, optics, superconductivity. We mention three classical versions of this method as a simple way for construction of particular solutions to those nonlinear Partial Differential Equations (PDE) which are integrable by the Inverse Spectral Transform (IST) (so called completely integrable PDE). First one is Zakharov-Shabat dressing method [1, 2], which uses integral Volterra operators. In particular, it solves initial value problem. Second version, $\bar{\partial}$-dressing method [3, 4, 5], is based on the Fredholm operators. This algorithm is simpler in comparison with Zakharov-Shabat dressing method, however, the manifold of particular solutions is more restricted. Different properties of integrable PDE have mostly evident interpretation in terms of this method (Miura and Bäclund transformations, commuting flows, relations among different hierarchies of equations). Finally, one should refer to Sato approach to the integrability [6], which is based on properties of pseudo-differential operators. Two last versions have been used for development of the new version of the dressing method, applicable to wide class of nonlinear PDE, which are not integrable by IST [7, 8]. Some applications of these equations has been outlined.

It is interesting to note that this new version of the dressing method can be readily written in Fourier form, Sec.2. Advantage of this form is evident in work with integral-differential equations. Below we consider an example leading to extension of Intermediate Long Wave

equation (ILW) into (2+1)-dimensions:

$$v_t - Tv_{x_1 x_1} - (v^2)_{x_1} = 0, \quad Tv(x_1, x_2) \equiv \frac{\pi}{\delta} \int_{-\infty}^{\infty} \coth\left(\frac{\pi}{2\delta}(\xi - x_2)\right) v(x_1, \xi) d\xi = 0. \tag{1}$$

Reduction $\partial_{x_2} = \partial_{x_1}$ transforms this equation into (1+1)-dimensional ILW [9, 10, 11, 12]. We provide overdetermined linear system for the eq. (22). Then we give the algorithm for construction of solutions (Sec.3) and derive formular for soliton. Then we consider limits of large and small δ (Sec.4), and figure out the asymptotic integrability in the later case [13]. Finally we give some remarks (Sec.5).

Note, that eq.(1) is integrable in classical sence, since it is compatibility condition for some overdetermined linear system. We don't consider other types of equations in this paper, which is aimed on the description of the algorithm for the dressing procedure in Fourier form.

2   Dressing method

We consider two-dimensional x-space, $x = (x_1, x_2)$, and time, t. Real variables x_1 and x_2 are related with complex spectral parameters k_1 and k_2, forming two-dimensional spectral space $k = (k_1, k_2)$ due to Fourier transform

$$f(x) = \int_{D_k} f(k) e^{i(k_1 x_1 + k_2 x_2)} dk_1 dk_2, \tag{2}$$

where D_k is 2-dimensional, either complex or real (k_1, k_2)-space, and $f(k) \to 0$ as $k \to \infty$. Thus this transformation maps space of functions of two complex variables into the space of functions of two real variables. This means that inverse transform may not be unique. This fact will not been discussed hereafter since inverse transform will not been used in this paper.

Our algorithm is based on the following integral equation

$$\Phi(k) \equiv \Phi(\lambda, \mu; k) = \int_{D_\nu} d\nu \int_{D_q} \Psi(\lambda, \nu; k - q) U(\nu, \mu; q) dq \equiv \Psi(k) * U(k), \tag{3}$$

which should be uniquely solvable for U: $U = \Psi^{-1} * \Phi$. Here we use another type of parameters denoted by Greek. In general these parameters are vector [8] but in our consideration they are just complex parameters; D_ν is complex ν-plane. We use $*$ to abbreviate combination of two types of integrals: integral with respect to "inside" Greek parameter (ν in eq.(3)) and over space of spectral parameters ($q = (q_1, q_2)$ in eq.(3)). In general

$$f(k) * g(k) = \int_{D_\nu} d\nu \int_{D_q} f(\mu_1, \ldots, \nu; k - q) g(\nu, \ldots; q) dq. \tag{4}$$

To avoid unambiguity we will write explicitly dependence on k while omit dependence on Greek parameters from notations in most of the cases.

Of principal meaning is how to introduce the dependence on parameters x (or k) and t. This defines particular nonlinear system of PDE which will be derived in result.

Let

$$k_1\Psi(k) = \Phi(k) * c(k) + k_1\delta(k), \tag{5}$$

$$e^{-2\delta k_2}\Psi(k) = \Phi(k) * \left(c(k)e^{-2\delta k_2}\right) + \Psi(k) + (e^{-2\delta k_2} - 1)\delta(k), \tag{6}$$

$$\Psi_t(k) = i(k_1\Phi(k)) * c(k) - i\Phi(k) * (k_1c(k)), \tag{7}$$

$$c(\lambda, \nu; k) = c_1(\lambda)c_2(\nu; k). $$

Being overdetermined system of equations for the function Ψ, it should be compatible, which leads to additional conditions for the functions Φ and c_2. To derive them let us multiply eqs. (5) and (6) by $e^{-2\delta k_2}$ and k_1 respectively and subtract one from another. The result is splitted into two equations:

$$\left(\left(k_1 - e^{-2\delta k_2}\right)\Phi(k)\right) * \left(c(k)e^{-2\delta k_2}\right) = 0, \tag{8}$$

$$\Phi(k) * \left((k_1 e^{-2\delta k_2} + 1)c_2\right) = 0. \tag{9}$$

Similarly, differentiating (5) with respect to t and using (7) we get two other conditions

$$\Phi_t(k) = ik_1^2\Phi(k), \tag{10}$$

$$c_{2t}(k) = -ik_1^2 c_2(k). \tag{11}$$

Eqs. (8-11) suggest the following form for Φ and c_2:

$$\Phi(k) = \Phi_0(k_1)e^{ik_1^2 t}\delta\left(e^{-2\delta k_2} - k_1\right), \tag{12}$$

$$c_2(k) = c_0(k_1)e^{-ik_1^2 t}\delta\left(e^{2\delta k_2} + k_1\right), \tag{13}$$

$$\tag{14}$$

where $\Phi_0(k)$ and $c_0(k)$ are arbitrary functions of argument. From another point of view equations (8,10) lead to nonlinear equation for U. In fact, due to (12) we can rewrite eq.(8) in the form

$$\left(k_1 - e^{-2\delta k_2}\right)\Phi(k) = \Phi_0(k_1)e^{ik_1^2 t}\left(k_1 - e^{-2\delta k_2}\right)\delta\left(e^{-2\delta k_2} - k_1\right) \equiv F(k), \tag{15}$$

thus $g(k) * F(k) \equiv 0$ for any well defined function g. Now, let us substitute (3) for Φ in eqs. (15,10), use eqs. (5-7) and apply operator $\Psi^{-1}(k)$ from the left. This results in

$$U(k) * (c_1 c_2(k)) * U(k) + k_1 U(k) - U(k) * \left(c_1 c_2(k)e^{-2\delta k_2}\right) * (e^{-2\delta k_2}U(k)) - \tag{16}$$
$$e^{-2\delta k_2}U(k) = 0,$$

$$-iU_t(k) - 2U(k) * (k_1 c_1) * U(k) - 2U(k) * c_1 * (k_1 U(k)) - k_1^2 U(k) = 0. \tag{17}$$

Introduce variables

$$w(k) = c_2(k) * U(k) * c_1, \quad V(k) = c_2(k) * (k_1 U(k)) * c_1, \tag{18}$$

where

$$f(k) * g = \int_{D_\nu} f(\mu_1, \ldots, \nu; k)g(\nu, \ldots)d\nu. \tag{19}$$

3

Apply operators $c_2(k)$ and c_1 from the left and from the right respectively to the eqs.(16,17). We get

$$V(k)(1 - e^{-2\delta k_2}) = w(k) * ((e^{-2\delta k_2} - 1)w(k)) - (k_1 e^{-2\delta k_2})w(k), \qquad (20)$$

$$-iw(k)_t - 2k_1 V(k) + k_1^2 w(k) - 2w(k) * (k_1 w(k)) = 0. \qquad (21)$$

Deriving the first of the above equations we used the fact that $c_2(k) \sim \delta\left(e^{2\delta k_2} + k_1\right)$ which identifies $c_1 e^{2\delta k_2}$ with $-c_1 k_1$ in this context. Eliminating V and introducing $u = w(e^{-2\delta k_2} - 1)$ we end up with equation

$$iu_t - k_1^2 \coth(k_2\delta)u + k_1 \int\limits_{D_q} u(k-q)u(q)dq = 0. \qquad (22)$$

This equation can be considered as compatibility condition of appropriate linear overdetermined system, which may be readily derived. In fact, apply the operator c_1 to the eqs. (16,17) from the right side. Resulting equations may be written in terms of the "spectral" function $\chi(k) = U(k) * c_1$:

$$k_1\chi(k) + \chi(k) * \left((1 - e^{-2\delta k_2})w(k)\right) - e^{-2\delta k_2}\chi(k) = 0, \qquad (23)$$

$$i\chi_t(k) = -2\chi(k) * (k_1 w(k)) - k_1^2\chi(k). \qquad (24)$$

This is linear system which we need. To check this statement, let us differentiate (23) with respect to t and use (24). We result in (22).

Transformation of the derived equations into x-form is straightforward if k_1 and k_2 are real and (2) is usual Fourier transform. Then x-form of $\coth(k_2\delta)$ is $\frac{-i\pi}{\delta}\coth\left(\frac{\pi x_2}{2\delta}\right)$ and we receive eq. (1). In general case of complex k equation (22) is better to multiply by $\left(1 - e^{-2\delta k_2}\right)$. Then we may formally write it in x-form using shift operator $\mathcal{D}f(x_1, x_2) = f(x_1, x_2 + 2i\delta)$:

$$(1 - \mathcal{D})\left(iu_t - (u^2)_{x_1}\right) + (1 + \mathcal{D})u_{x_1 x_1} = 0. \qquad (25)$$

We can write the above linear system (23,24) in x-form using transformation (2):

$$i\chi_{x_1}^+ + \chi^+ u + \chi^- = 0, \quad \chi^-(x_1, x_2) = \chi^+(x_1, x_2 + 2i\delta), \qquad (26)$$

$$i\chi_t^+ - 2i\chi^+ w_{x_1} - \chi_{x_1 x_1}^+ = 0, \qquad (27)$$

$$u(x_1, x_2) = w(x_1, x_2 + 2i\delta) - w(x_1, x_2), \quad w = -u - iTu.$$

After reduction $\partial_y = \partial_x$ this system becomes linear overdetermined system for ILW [10].

3   Solutions

Find Ψ from the eq.(5) and substitute it into the eq.(3). Thus we receive

$$\Phi(k) = \tilde{\Psi}(k) * U(k) + U(k), \quad \tilde{\Psi}(k) = \frac{1}{k_1}(\Phi(k) * c(k)), \qquad (28)$$

where

$$\Phi(k) \equiv \Phi(\lambda, \mu; k) = \Phi_0(\lambda, \mu; k_1)e^{ik_1^2 t}\delta\left(e^{-2\delta k_2} - k_1\right), \qquad (29)$$

$$c(k) \equiv c_1(\lambda)c_2(\mu; k), \quad c_2(k) \equiv c_2(\mu; k) = c_0(\mu; k_1)e^{-ik_1^2 t}\delta\left(e^{2\delta k_2} + k_1\right). \qquad (30)$$

4

Thus

$$\tilde{\Psi}(k) \equiv \tilde{\Psi}(\lambda, \mu; k) = \int\limits_{D_\nu} \frac{d\nu}{k_1} \left(\Phi_0(\lambda, \nu; q)e^{iq^2 t}\right)\Big|_{q=\frac{-k_1 e^{-2\delta k_2}}{1-e^{-2\delta k_2}}} \left(c_1(\nu)c_2(\mu; q)e^{-iq^2 t}\right)\Big|_{q=\frac{k_1}{1-e^{-2\delta k_2}}} \quad (31)$$

Hereafter it is convenient to represent the above equation (28) in x-form:

$$\Phi(\lambda, \mu; x) = \int\limits_{D_\nu} \tilde{\Psi}(\lambda, \nu; x)U(\nu, \mu; x)d\nu + U(\lambda, \mu; x). \quad (32)$$

Solving (32) for U we calculate $w(x) = \int\limits_{D_\nu} \int\limits_{D_\mu} c_2(\mu; x)U(\mu, \nu; x)c_1(\nu)$ and $u(x) = w(x_1, x_2 + 2\delta i) - w(x)$. Note, that equation (32) can be solved exactly in particular case of degenerate functions $\tilde{\Psi}(\lambda, \mu; x) = \sum_j \psi_{1j}(\lambda; x)\psi_{2j}(\mu; x)$.

In the simplest case let $c_1(\lambda) = \delta(\lambda)$, $\Phi_0(\lambda, \mu; k_1) = r_1(\lambda, \mu)\delta(k_1 - a)$, $c_0(\mu; k_1) = r_2\delta(\mu)\delta(k_1 - b)$. Then one can easy receive

$$\Phi(\lambda, \mu; x) = \tilde{\Psi}(\lambda, 0; x)U(0, \mu; x) + U(\lambda, \mu; x), \quad (33)$$

where $\tilde{\Psi}(\lambda, 0; x)$ is x-form of the function

$$\tilde{\Psi}(\lambda, 0; k) = \frac{1}{k_1}\Phi_0(\lambda, 0; q)e^{iq^2 t}\Big|_{q=\frac{-k_1 e^{-2\delta k_2}}{1-e^{-2\delta k_2}}} r_2\delta(q - b)e^{-iq^2 t}\Big|_{q=\frac{k_1}{1-e^{-2\delta k_2}}}.$$

One has

$$U(0, \mu; x) = \frac{\Phi(0, \mu; x)}{\tilde{\Psi}(0, 0; x) + 1}. \quad (34)$$

Then we get line soliton solution of (22) ($r_1(0, 0) = r_1$):

$$\begin{aligned}
u &= \frac{1}{R_1 + R_2 \cosh(\eta)}, \quad (35)\\
R_1 &= \frac{\alpha}{2\beta^2}, \quad R_2 = \frac{\sqrt{\alpha^2 + \beta^2}}{2\beta^2},\\
\eta &= -2\beta t_1 + \frac{\arctan(\beta/\alpha)}{\delta}t_2 + 4\alpha\beta t,\\
a &= \alpha + i\beta, \quad b = -\alpha + i\beta, \quad r_1 r_2 = 2i\beta e^{-i\arctan(\beta/\alpha)}.
\end{aligned}$$

4 Limits $\delta \to \infty$ and $\delta \to 0$

As we mansion above, the reduction $\partial_x = \partial_y$ transforms eq.(1) into (1+1)-dimensional ILW, which in turn has two reductions leading to appropriate completely integrable systems: $\delta \to \infty$ (Benjamin-Ono equation (BO) [14, 15]) and $\delta \to 0$ (Korteweg-de Vries equation (KdV) [16]). We consider these reductions in application to our (2+1)-dimensional equation. We will see that limit $\delta \to \infty$ results in another completely integrable system with Hilbert operator instead of T-operator. But limit $\delta \to 0$ does not lead to completely integrable system. Instead we get the system, which is integrable in "asymptotic" sence [13].

First, we rewrite this system in terms of functions $\psi^{\pm}$, defined by $\chi^{+} = e^{i(fx_1 + gx_2 + ht)}\psi^{+}$, and use variable τ instead of t: $\partial_t = \partial_\tau + \alpha\partial_{x_1}$. One has

$$i\psi^{+}_{x_1} + \psi^{+}(u - \lambda) = \mu\psi^{-}, \qquad (36)$$

$$i\psi^{+}_{\tau} - 2i(\lambda + 1/(2\delta))\psi^{+}_{x_1} - \psi^{+}_{x_1 x_1} - (2iw_{x_1} + \nu)\psi^{+} = 0, \qquad (37)$$

$$\alpha = -\frac{1}{\delta}, \quad h = \lambda^2 + \nu, \quad f = \lambda = -k\coth(2k\delta), \quad e^{-2g\delta} = -\mu = -k\operatorname{cosech}(2k\delta),$$

1. Let $\delta \to \infty$. Then operator T in (1) transforms into Hilbert operator

$$T(f) \simeq 2\int\limits_{-\infty}^{\infty} \frac{d\xi}{\xi - x_2} f(\xi)$$

and one should take $\mu = 2k$, $\lambda = -k$ in eq.(36,37). Functions $\psi^{\pm}$ are analytical in upper and lower half of complex z plane: $Re(z) = x$. If $\partial_{x_1} = \partial_{x_2}$, then our nonlinear equation becomes BO.

2. Consider the limit $\delta \to 0$. Let $u = \delta U$, $\delta \ll 1$, and expand the above linear system in the series up to δ^2. One has:

$$i(\psi_x - \psi_y) + \delta(\psi_{yy} + \psi U + k^2\psi) + \frac{2i\delta^2}{3}\left(\psi_{yyy} + k^2\psi_y\right) = 0, \qquad (38)$$

$$i\psi_\tau - \psi_{yy} - \psi(2iw - \nu) - i\delta\left(2\psi_{yyy} + \frac{2}{3}\psi_y(3U + k^2) + \psi(U_x + U_y)\right) + \qquad (39)$$

$$\frac{\delta^2}{3}\left(7\psi_{yyyy} + 6\psi_{yy}(U + k^2) + 6\psi_y U_y + \psi(3U_{yy} + 3U^2 + 2k^2 U - k^4)\right) = 0$$

This system is compatible up to δ^2 if

$$\delta v_t - v_x + \partial_y^{-1} v_{xx} - \frac{\delta^2}{3} v_{xyy} + 4\delta^2 v v_x = 0, \quad v = w_y, \quad U = 2i(w_y + \delta i w_{yy}). \qquad (40)$$

We conclude that derived equation (40) provides compatibility of related overdetermined system only up to the order δ^2. Thus this equation can be treated as "asymptotically" integrable [13].

5   Remarks

Although we have considered only completely integrable example, Fourier form of the dressing method can be expanded on other types of systems, discussed in [8]. Their physical application is one of the mostly interesting problems now. One should note that similar "symbolic" approach to nonlinear PDE has been used by other authors in different contexts [17, 18, 19, 20]. The Lax pair for PDE with quadratic nonlinearity in Fourier form is analyzed in [20] (compare with eqs.(23,24)).

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